Continua and dimension

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Abstract. We briefly discuss the relationship between chainability and dimension and describe a non-metric chainable continuum of inductive dimensions 2. We also give topological characterizations of selected hyperspaces of infinite dimensional compacta in the Hilbert cube.

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1. Introduction

Although the development of both dimension theory and continuum theory began in the early XXth century, even nowadays new results are obtained. We would like to present a few facts from the border of this two fields.

In the paper all *spaces* are Hausdorff and a *continuum* is a non-empty, compact connected space.

Let A, B be disjoint closed subsets of a space X. Suppose that C is a closed subset of X and $C \subset X \setminus (A \cup B)$. Then C is called a *separator* between A and B if there are disjoint open subsets U, V of X such that $A \subset U$, $B \subset V$ and $X \setminus C = U \cup V$; and C is called a *cut* between A and B provided that every continuum that meets both A and B meets C. Notice that every separator is a cut. If X is a compact, metric and locally connected space, then a closed subset $C \subset X$ cuts X between A and B if and only if C is a separator in X between A and B [13, Theorem 1, p. 238].

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The definitions of the inductive dimension functions Ind and Dg are similar: $\operatorname{Ind} X$ or $\operatorname{Dg} X$ equals -1 iff $X = \emptyset$. For a non-empty normal space X, $\operatorname{Ind} X$ ($\operatorname{Dg} X$, respectively) is the smallest non-negative integer n such that between any pair of disjoint closed subsets A and B of X, there is a separator (cut, respectively) C with $\operatorname{Ind} C$ ($\operatorname{Dg} C$, respectively) $\leqslant n-1$, provided that such a number n exists. If there is no such n then $\operatorname{Ind} X$ ($\operatorname{Dg} X$, respectively) $= \infty$. Assuming that the set A in the above definition of Ind is a singleton we obtain the definition of the dimension function ind. Transfinite extensions of the above dimension functions are obtained in the usual manner and denoted by trDg, trind and trInd.

It is known that if X is a separable metric space then $\operatorname{ind} X = \operatorname{Ind} X = \dim X$. In particular, for metric continua and subspaces of I^∞, where $I = [0, 1]$, all dimension functions coincide.

A space X is *strongly infinite dimensional* if there exists a sequence $(A_n, B_n)_n$ of closed disjoint subsets of X such that for each sequence $(C_n)_n$ of closed separators of X between A_n and B_n we have $\bigcap_n C_n \neq \emptyset$.

A space is *weakly infinite dimensional* if it is not strongly infinite-dimensional.

A space is *strongly countable dimensional* if it is a countable union of closed finite dimensional subsets.

By a *Hilbert cube* we mean a homeomorphic copy of I^∞. The Hilbert cube is strongly infinite dimensional and not strongly countable dimensional.

2. Chainable continua and dimension

A *chain* is a finite collection of sets $U_1, \dots, U_n$ such that $U_i \cap U_j \neq \emptyset$ iff $|i-j| \leqslant 1$. A non-empty normal space X is said to be *chainable* if every open cover of X can be refined by an open (or equivalently, closed) chain. Is is easy to see that any chainable space X is a continuum, and $\dim X = 1$ unless X is a single point. We say that a point x of a chainable space X is an *end point* if every open cover of X can be refined by an open (or equivalently closed) chain $\{U_1, \dots, U_n\}$ such that $x \in U_1$.

In a sense, chainable continua are the simplest ones among continua of covering dimension 1. It was an open question if every chainable continuum is one dimensional in the sense of dimensions ind, Ind and Dg. In 1959 Mardešič [16] constructed the first chainable continuum X with $\operatorname{ind} X = \operatorname{Ind} X = 2$ and thus answered this question in the negative.[1] Later in 1963 Pasynkov [17] strengthened Lokucievskiĭ's [14] counterexample to a sum theorem for Ind. He has constructed a chainable continuum X with $\operatorname{Ind} X = 2$ such that X is the union of two chainable subcontinua one dimensional in all sense.

Bobkov [1] constructed in 1979 the first known first-countable chainable continuum X with $\operatorname{ind} X = 2$, and later Chatyrko [4] in 1990 gave, for every $n \in \mathbb{N}$, examples of first-countable chainable continua X_n such that $\operatorname{ind} X_n = n$. The spaces X_{n+1} and X_n are linked by natural projections $f_n \colon X_{n+1} \to X_n$ which lead to an inverse sequence whose limit space X_∞ is a chainable continuum such that every proper subcontinuum of X_∞ has infinite inductive dimension ind. Recently Charalambous

[1] Lunc [15] constructed the first known example of a continuum with non-coinciding dimensions.

and Krzempek [3], for each pair of ordinals α,β with $1 \leqslant \alpha \leqslant \beta \leqslant \omega(\mathfrak{c}^+)$, where $\omega(\mathfrak{c}^+)$ is the first ordinal of cardinality $\mathfrak{c}^+$, have presented first-countable chainable continua $S_{\alpha,\beta}$ such that $\operatorname{trDg} S_{\alpha,\beta} = \alpha$ and $\operatorname{trind} S_{\alpha,\beta} = \operatorname{trInd} S_{\alpha,\beta} = \beta$.

In this chapter we construct a chainable continuum $K(A)$ with $\operatorname{Ind} K(A) = 2$ and then we describe how to obtain chainable continua $Q(A,B)$ such that $\operatorname{Ind} Q(A,B) = 2$ and $Q(A,B)$ is the union of two continua which are chainable and one-dimensional with respect to ind and Ind. The construction described below is a modification of the ones of Chatyrko [5] and Pasynkov [17]. Krzempek noticed that one can combine techniques from [17] and [5] in order to obtain a new class of examples. He suggested this method to the first named author.

Denote by $\omega(\mathfrak{c})$ the first ordinal of cardinality $\mathfrak{c}$, by W (W^0) the set of all ordinals $< \omega(\mathfrak{c})$ ($\leqslant \omega(\mathfrak{c})$, respectively) and by L the long segment (cf. [9, Example 2.2.13]) of length $\omega(\mathfrak{c})$. If $x \in L$ then $x = \alpha + t$ where $\alpha < \omega(\mathfrak{c})$ and $t \in [0,1)$ (for convention if $t = 0$ then we will simply write α instead $\alpha + 0$) or $x = \omega(\mathfrak{c})$. Denote by Ω the product space $L \times [0,1]$. For $\alpha \in W^0$ put $I_\alpha = \{\alpha\} \times I \subset \Omega$. For each $\alpha \in W$ fix a homeomorphism $h_\alpha\colon [\alpha, \alpha+1] \to [0,1]$ such that $h_\alpha(\alpha) = 0$ and $h_\alpha(\alpha+1) = 1$. For any $0 < a \leqslant b < 1$ denote by $[a,b]_\alpha$ the subspace $h_\alpha^{-1}([a,b])$.

Fix a non-empty set $A \subset (0,1)$ and write $\mathcal{S}_A$ for the family of all sequences $\{x_n\}_{n=0}^\infty$ such that:

(1) $x_n \in (0,1)$ for all $n \in \mathbb{N}$,
(2) $\lim_{n\to\infty} x_n = x_0 \in A$,
(3) subsequence $\{x_{2k-1}\}_{k=1}^\infty$ is strictly monotonically increasing and
(4) subsequence $\{x_{2k}\}_{k=1}^\infty$ is strictly monotonically decreasing.

Consider any function $\phi\colon W \to \mathcal{S}_A$ such that $\operatorname{card} \phi^{-1} x = \mathfrak{c}$ for every $x \in \mathcal{S}_A$.

For $\alpha \in W$ denote by $F_\alpha(A)$ the subspace of $[\alpha, \alpha+1] \times [0,1]$ which is the union of

- segments $[0, 1/3]_\alpha \times \{0\}$ and $[2/3, 1]_\alpha \times \{1\}$,
- segments $[1/3, 2/3]_\alpha \times \phi(\alpha)$,
- all segments which connect points $(\alpha + 1/3, 0)$ and $(\alpha + 2/3, x_1)$, $(\alpha + 1/3, x_1)$ and $(\alpha + 2/3, x_3)$,... and all segments which connect points $(\alpha + 2/3, 1)$ and $(\alpha + 1/3, x_2)$, $(\alpha + 2/3, x_2)$ and $(\alpha + 1/3, x_4)$, ..., where $\lim_{n\to\infty} x_n = x_0$ and $\{x_n\}_{n=0}^\infty = \phi(\alpha)$.

From definition of $F_\alpha(A)$ we get the following proposition.

Proposition 2.1. *$F_\alpha(A)$ is a chainable continuum for each $\alpha \in W$.*

In order to show the next statement we will need the following trivial proposition.

Proposition 2.2. *Suppose that $X = X_1 \cup X_2$, where X_1 and X_2 are chainable continua, $x_1, x_2 \in X_1$ are endpoints of X_1, $x_2, x_3 \in X_2$ are endpoints of X_2 and $X_1 \cap X_2 = \{x_2\}$. Then X is a chainable continuum with endpoints x_1, x_3.*

Define

$$
\begin{aligned}
K_0(A) &= \{(0,0)\},\\
K_{\alpha+1}(A) &= K_\alpha(A) \cup F_\alpha(A) \cup I_{\alpha+1},\\
K_\alpha(A) &= I_\alpha \cup \bigcup_{\beta<\alpha} K_\beta(A), \text{ if } \alpha \text{ is a limit ordinal, and}\\
K(A) &= K_{\omega(\mathfrak{c})}(A).
\end{aligned}
$$

Lemma 2.3. *For every $\alpha \in W^0$ the space $K_\alpha(A)$ is a chainable continuum with end points $(0,0)$ and $(\alpha,0)$.*

Proof. It is easily seen that $K_0(A)$ is a chainable continuum, and by Proposition 2.2 if $K_\alpha(A)$ is a chainable continuum with end points $(0,0)$ and $(\alpha,0)$ then the statement holds for $K_{\alpha+1}(A)$.

Suppose that α is a limit ordinal and $K_\beta(A)$ is a chainable continuum for every $\beta<\alpha$ with end points $(0,0)$ and $(\beta,0)$. Consider any open cover $\mathcal{U}$ of $K(\alpha)$. We can refine open cover $\{U\cap I_\alpha : U\in\mathcal{U}\}$ of I_α by a closed chain $\{G_1,\ldots,G_n\}$ such that $(\alpha,1)\in G_1\setminus G_2$ and $(\alpha,0)\in G_n\setminus G_{n-1}$. Put $G_0=\{(\alpha,1)\}$ and $G_{n+1}=\{(\alpha,0)\}$. Sets $G_0,\ldots,G_{n+1}$ swell to a closed chain $\{\operatorname{cl}V_0,\ldots\operatorname{cl}V_{n+1}\}$ which refines $\mathcal{U}$ where $V_0,\ldots,V_{n+1}$ are open in $K_\alpha(A)$. There exists an ordinal number $\beta_0<\alpha$ such that $K_\alpha(A)$ *cap*$([\beta_0,\alpha]\times\{1\})\subset V_1\cap V_0$. Consider a set H open in subspace $[0,\alpha]\subset L$ where

$$H=(\beta_0,\alpha]\setminus\pi\Big(K_\alpha(A)\setminus\bigcup_{i=1}^{n}V_i\Big)$$

and $\pi\colon L\times[0,1]\to L$ is the projection. Notice that there exists an ordinal number $\beta\in(\beta_0,\alpha)$ such that $(\beta,\alpha]\subset H$. By Proposition 2.2 for $K_\beta(A)\cup F_\beta(A)$, there exist a closed chain $\{F_1,\ldots,F_m\}$ which refines $\mathcal{U}$ such that $(0,0)\in F_1\setminus F_2$ and $(\beta+1,1)\in F_m\setminus F_{m-1}$. Put

$$F_{m+i}=\operatorname{cl}V_i\cap\pi^{-1}[[\beta+1,\alpha]] \text{ for } i=1,\ldots,n.$$

It is easily seen that $F_1,\ldots,F_{m+n}$ is a closed chain which refines $\mathcal{U}$ and covers the space $K_\alpha(A)$ such that $(0,0)\in F_1\setminus F_2$ and $(\alpha,0)\in F_{m+n}\setminus F_{m+n-1}$. □

Corollary 2.4. *$K(A)$ is a chainable continuum.*

The following statement is clear.

Proposition 2.5. *If $U\subset\Omega$ is an open set then there exists an ordinal number $\alpha<\omega(\mathfrak{c})$ such that*

$$[\alpha,\omega(\mathfrak{c})]\times\pi_{[0,1]}[U\cap I_{\omega(\mathfrak{c})}]\subset U,$$

where $\pi_{[0,1]}\colon L\times[0,1]\to[0,1]$ is the projection.

Lemma 2.6. *Suppose that $U_1,U_2\subset K(A)$ are disjoint open sets. If*

$$p\in A\cap\operatorname{cl}\pi_{[0,1]}[U_1\cap I_{\omega(\mathfrak{c})}]\cap\operatorname{cl}\pi_{[0,1]}[U_2\cap I_{\omega(\mathfrak{c})}],$$

then there exists an ordinal number $\alpha<\omega(\mathfrak{c})$ such that $[1/3,2/3]_\alpha\times\{p\}\subset K(A)\setminus(U_1\cup U_2)$.

Proof. By Proposition 2.5 there exists an ordinal number $\beta<\omega(\mathfrak{c})$ such that $(\beta,\omega(\mathfrak{c})]\times\pi_{[0,1]}[U_i\cap I_{\omega(\mathfrak{c})}]\subset U_i$ for $i=1,2$. Let $\{x_n^i\}_{n=1}^\infty\subset\pi_{[0,1]}[U_i\cap I_{\omega(\mathfrak{c})}]$ be a sequence in $\phi(W)$ such that $\lim_{n\to\infty}x_n^i=p$ for $i=1,2$. There exists an ordinal number $\alpha>\beta$ such that $\{x_n^i : n\in\mathbb{N} \text{ and } i=1,2\}\subset\phi(\alpha)$. Of course $[1/3,2/3]_\alpha\times\{x_n^i : n\in\mathbb{N}\}\subset U_i$ for $i=1,2$ and, by the definition of F_α, we have $[1/3,2/3]_\alpha\times\{p\}\subset\operatorname{cl}U_1\cap\operatorname{cl}U_2\subset K(A)\setminus(U_1\cup U_2)$. □

Theorem 2.7. $\operatorname{ind} K(A) = \operatorname{Ind} K(A) = 2$ *if and only if* $\operatorname{int} A \neq \emptyset$.

Proof. Suppose that $\operatorname{int} A \neq \emptyset$. Thus there exist $a, b \in A$ such that $[a, b] \subset A$. Put $x_1 = (\omega(\mathfrak{c}), a)$ and $x_2 = (\omega(\mathfrak{c}), b)$. Consider a separator P in the space $K(A)$ between the points x_1, x_2 and disjoint open sets U',V' of $K(A)$ such that $K(A) \setminus P = U' \cup V'$ and $x_1 \in U'$, $x_2 \in V'$. There exist disjoint open sets $U, V \subset \Omega$ such that $K(A) \setminus P = K(A) \cap (U \cup V)$ and $x_1 \in U$, $x_2 \in V$. If $\operatorname{int} \pi_{[0,1]}[P \cap I_{\omega(\mathfrak{c})}] \neq \emptyset$ then $\operatorname{ind} P \geqslant 1$. Suppose that $\operatorname{int} \pi_{[0,1]}[P \cap I_{\omega(\mathfrak{c})}] = \emptyset$. Thus there exists a point p such that

$$p \in [a, b] \cap \operatorname{cl} \pi_{[0,1]}[U \cap I_{\omega(\mathfrak{c})}] \cap \operatorname{cl} \pi_{[0,1]}[V \cap I_{\omega(\mathfrak{c})}].$$

By Lemma 2.6 there exists an ordinal number $\alpha < \omega(\mathfrak{c})$ such that

$$[1/3, 2/3]_\alpha \times \{p\} \subset \operatorname{cl} U \cap \operatorname{cl} V \subset K(A) \setminus (U \cup V) = P.$$

Thus $\operatorname{ind} P \geqslant 1$ and $\operatorname{ind} K(A) = \operatorname{Ind} K(A) = 2$.

Now suppose that $\operatorname{int} A = \emptyset$. It is enough to prove that $\operatorname{ind} K(A) \leqslant 1$. Using rectangular open sets it is easy to see that the statement holds. □

Remark 2.8. Notice that $X(A) = \bigcup_{\alpha \in W^0} I_\alpha \cup \bigcup_{\alpha \in W} [1/3, 2/3]_\alpha \times \phi(\alpha)$ is a closed subspace of $K(A)$. In order to prove Theorem 2.7 it is enough to show that $\operatorname{ind} X(A) = \operatorname{Ind} X(A) = 2$ if and only if $\operatorname{int} A \neq \emptyset$, but it follows from [5, Proposition 4.1].

Example 2.9. Let $A = (0, 1)$, then ,by Corollary 2.4, $K(A)$ is a chainable continuum and, by Theorem 2.7, we have $\operatorname{ind} K(A) = \operatorname{Ind} K(A) = 2$.

Consider product $\{0, 1\} \times L$ with the linear order $\leqslant$ such that $(a, \alpha) \leqslant (b, \beta)$ if and only if (1) $a < b$, (2) $a = b = 0$ and $\alpha \leqslant \beta$ or (3) $a = b = 1$ and $\beta \leqslant \alpha$ where $(a, \alpha), (b, \beta) \in \{0, 1\} \times L$. Denote by E the upper semi-continuous decomposition of $\{0, 1\} \times L$ into the set $X = \{(0, \omega(\mathfrak{c})), (1, \omega(\mathfrak{c}))\}$ and singletons of $\{0, 1\} \times L \setminus X$. The quotient space $M = (\{0, 1\} \times L)/E$ is a linearly ordered continuum. For convention, if $q \colon \{0, 1\} \times L \to M$ is the quotient mapping then we will denote points $q(0, \alpha)$, $q(1, \alpha)$ by α^0, α^1 respectively. Notice that $L^0 = q[\{0\} \times L]$ and $L^1 = q[\{1\} \times L]$ are homeomorphic to L. There exists a continuous symmetry $\sigma \colon M \to M$ such that

(1) $\sigma(0^0) = 0^1$ and $\sigma(0^1) = 0^0$,
(2) $\sigma \circ \sigma = \mathrm{id}$ and
(3) if $x, y \in M$ and $x \leqslant y$, then $\sigma(y) \leqslant \sigma(x)$.

Consider product space $M \times [0, 1]$ and homeomorphism $\rho \colon M \times [0, 1] \to M \times [0, 1]$ given by the formula

$$\rho(x, t) = (\sigma(x), 1 - t) \text{ for } (x, t) \in M \times [0, 1].$$

Then the sets $L^i \times [0, 1]$ for $i = 0, 1$ are homeomorphic to Ω. Thus previously constructed chainable continuum $K(A)$ can be considered as a subset of $L^0 \times [0, 1]$. For a set $B \subset (0, 1)$ denote by $K'(B)$ the chainable continuum $\rho[K(1 - B)]$ where $1 - B = \{1 - b \colon b \in B\} \subset (0, 1)$. Put $Q(A, B) = K(A) \cup K'(B)$. The following theorem is clear.

Theorem 2.10. *$Q(A, B)$ is a chainable continuum and* $\operatorname{ind} Q(A, B) = \operatorname{Ind} Q(A, B) = 2$ *if and only if* $\operatorname{int}(A \cup B) \neq \emptyset$.

Example 2.11. Let $A = \mathbb{Q} \cap (0,1)$ and $B = (0,1) \setminus A$. Then, by Theorem 2.10, $Q(A,B)$ is a chainable continuum and $\operatorname{ind} Q(A,B) = \operatorname{Ind} Q(A,B) = 2$. Notice that $Q(A,B)$ is the union of chainable continua $K(A)$ and $K'(B)$ (cf. Corollary 2.4) which are one-dimensional with respect to ind and Ind (cf. Theorem 2.7).

3. Infinite dimensional compact subsets of the Hilbert cube

In this chapter we would like to recommend a few recent results, obtained with the method of absorbers and concerning hyperspaces whose definitions refer to the dimension of the elements. Theory of absorbers gives a topological characterization of some incomplete spaces. As an example of its application we present the proof of the theorem stating that strongly countable dimensional compacta of positive dimension form a space homeomorphic to the Hurewicz set.

Let (X,d) be a metric space. Denote by 2^X the space of all nonempty compact subsets of X equipped with the Hausdorff metric

$$d_H(K,L) = \inf\{\epsilon > 0 : K \subset B(L;\epsilon) \quad \text{and} \quad L \subset B(K;\epsilon)\},$$

where $B(A;\epsilon)$ stands for the open ϵ-ball about the subset A in X. Denote by $C(X)$ the subspace of 2^X consisting of all nonempty continua in X.

By a *hyperspace* of X we mean a subspace of 2^X.

If X is a locally connected nondegenerate continuum without free arcs, then 2^X and $C(X)$ are Hilbert cubes [6]. In particular, 2^{I^∞} and $C(I^\infty)$ are homeomorphic to I^∞.

Let X be a Hilbert cube with a metric d. Recall that a closed subset B of X is a *Z-set* in X if B satisfies the following condition:

$$\begin{aligned}&\text{for any } \epsilon > 0 \text{ there exists a continuous mapping } f : X \to X \text{ such that}\\ &f(X) \cap B = \emptyset \quad \text{and} \quad \widetilde{d}(f, \mathrm{id}_X) = \sup\{d(f(x),x) : x \in X\} < \epsilon.\end{aligned} \tag{Z}$$

A subset $B \subset X$ is called a *σZ-set in X* if B is the countable union of Z-sets in X. Observe that B is a σZ-set in X if and only if B is an F_σ-set in X and condition (Z) holds.

Let $\mathcal{M}$ be a class of spaces which is topological (i.e., if $M \in \mathcal{M}$ then each homeomorphic image of M belongs to $\mathcal{M}$) and closed hereditary (i.e., each closed subset of $M \in \mathcal{M}$ is in $\mathcal{M}$).

A subset A of a Hilbert cube X is a *$\mathcal{M}$-absorber in X* provided that

1. $A \in \mathcal{M}$;
2. A is contained in a σZ-set in X;
3. A is strongly $\mathcal{M}$-universal, i.e., for each subset $M \in \mathcal{M}$ of I^∞ and for each compact set $K \subset I^\infty$, any embedding $f : I^\infty \to X$ such that $f(K)$ is a Z-set in X can be approximated arbitrarily closely (in the "sup" metric $\widetilde{d}$) by an embedding $g : I^\infty \to X$ such that $g(I^\infty)$ is a Z-set in X, $g|K = f|K$ and $g^{-1}(A) \setminus K = M \setminus K$.

If $\mathcal{M}$-absorbers in a Hilbert cube exist then they are unique up to homeomorphisms.

Lemma 3.1 ([8]). *If $A \subset X$ and $B \subset Y$ are $\mathcal{M}$-absorbers in Hilbert cubes X and Y then there exists a homeomorphism $h : X \to Y$ such that $h(A) = B$. Moreover, if $X = Y$ then h can be chosen arbitrarily close to the identity.*

Verifying strong $\mathcal{M}$-universality of a subset of a Hilbert cube is usually difficult. The following lemma, using techniques from [8], may simplify this task.

Lemma 3.2 ([18]). *Assume that a family $\mathcal{A} \subset 2^{I^n}$, $n \in \mathbb{N} \cup \{\infty\}$ is topological and $\mathcal{A} \in \mathcal{M}$. Let $n \in \mathbb{N} \cup \{\infty\}$, $n \geq 2$, $\mathcal{S} \in \{C(I^n), 2^{I^n}\}$. Suppose $\mathcal{A}$ is a subset of $\mathcal{S}$ such that*

1. *$\mathcal{A} \in \mathcal{M}$,*
2. *for an arbitrary set $M \subset I^\infty$, $M \in \mathcal{M}$, there exists a continuous mapping $\xi : I^\infty \to \mathcal{S}$ such that $\xi^{-1}(\mathcal{A}) = M$ and*
3. *there exists an embedding $\theta : I^\infty \to C([-1,1]^\infty)$ such that for every $\epsilon \in (0, \frac{1}{2}]$, for each graph $\Gamma \in \mathcal{S}$, $\Gamma \subset [\epsilon, 1-\epsilon]^n$ with straight line edges, for each nonempty subset T of vertices of Γ and for each continuum $C \in \theta(I^\infty)$, the union*

$$\Gamma \cup \bigcup_{v \in T} (v + \epsilon C) \cup \bigcup_{v \in T} (v + \epsilon \xi(x))$$

belongs to $\mathcal{A}$ if and only if $x \in M$.

Then $\mathcal{A}$ is strongly $\mathcal{M}$-universal in $\mathcal{S}$.

Different Borel or desciptive classes satisfy the conditions imposed on a class $\mathcal{M}$ in the definition of $\mathcal{M}$-absorbers. These classes play an important role in investigating topological structures of incomplete spaces. It is known that for all Borel classes, with exception of G_δ's and lower classes, absorbers in the Hilbert cube do exist. Moreover, absorbers belonging to different descriptive classes are pairwise not homeomorphic.

Notice that F_σ-absorbers are, in particular, σZ-sets. A standard example of an F_σ-absorber in I^∞ is its *pseudoboundary* $B(Q) = I^\infty \setminus (0,1)^\infty$.

Theorem 3.3 ([7]). *1. If $n \geq 1$ then the hyperspace $D^{\geq n}(I^\infty)$ of all compacta of dimension $\geq n$ is an F_σ-absorber in 2^{I^∞}.*
2. For $n \geq 2$ the hyperspace $D^{\geq n} \cap C(I^\infty)$ of all continua of dimension $\geq n$ is an F_σ-absorber in $C(I^\infty)$.
3. All infinite dimensional compacta in I^∞ form an $F_{\sigma\delta}$-absorber in 2^{I^∞}.

The other example of an F_σ-absorber is the hyperspace of all decomposable subcontinua of the cube I^k, $k \in \mathbb{N} \cup \{\infty\}$, considered as a subset of the Hilbert cube $C(I^k)$ [19]. Recall that a continuum is *decomposable* provided that it is the union of two proper subcontinua.

Let Π^1_1 denote the class of coanalytic sets. The Π^1_1-absorbers are also called *coanalytic aborbers*. A standard example of a Π^1_1-absorber in the Hilbert cube 2^I is the Hurewicz set $\mathcal{H}$ consisting of all nonempty countable closed subsets of the unit interval I [2]. By lemma 3.1 every Π^1_1-absorber in a Hilbert cube is homeomorphic to $\mathcal{H}$ and its complement is homoeomorphic to $2^I \setminus \mathcal{H}$.

Theorem 3.4 ([12]). *Denote by $\mathcal{SCD}_k(I^\infty)$ and $\mathcal{SCD}_k \cap C(I^\infty)$ the hyperspaces of all strongly countable dimensional compacta and continua in I^∞ of dimension at least k, considered as subspaces of 2^{I^∞} and $C(I^\infty)$, respectively. For every positive number*

k the hyperspaces $\mathcal{SCD}_k(I^\infty)$ and $\mathcal{SCD}_{k+1} \cap C(I^\infty)$ are Π^1_1-absorbers in the Hilbert cubes 2^{I^∞} and $C(I^\infty)$, respectively.

Proof. The families $\mathcal{SCD}_k(I^\infty)$ and $\mathcal{SCD}_{k+1} \cap C(I^\infty)$ are coanalytic [10]. Obviously $\mathcal{SCD}_k(I^\infty) \subset D^{\geq 1}(I^\infty)$ and $\mathcal{SCD}_{k+1} \cap C(I^\infty) \subset D^{\geq 2} \cap C(I^\infty)$ and thus they are contained in appropriate σZ-sets (Theorem 3.3).

To verify Π^1_1-universality we use Lemma 3.2. Let M be a coanalytic set in I^∞. First we have to construct a continuous mapping $\xi : I^\infty \to C(I^\infty)$ such that $\xi^{-1}(\mathcal{SCD}_k(I^\infty)) = M$.

Let $\mathbb{N}^\mathbb{N}$ be the Baire space of all infinite sequences of natural numbers. The set $\mathbb{N}^{<\mathbb{N}}$ of all (nonempty) finite sequences of natural numbers can be considered as a subspace of $\mathbb{N}^\mathbb{N}$. For a sequence $\sigma \in \mathbb{N}^{<\mathbb{N}} \cup \mathbb{N}^\mathbb{N}$ denote its length by $|\sigma|$ (if $\sigma \in \mathbb{N}^\mathbb{N}$ then $|\sigma| = \infty$) and the finite sequence of the first n elements ($n \leqslant |\sigma|$) by $\sigma \restriction n$. If $|\sigma| < |\sigma'|$ and $\sigma' \restriction |\sigma| = \sigma$ then we write $\sigma < \sigma'$.

By [2, Lemma 1.5] there exists a function $S : \mathbb{N}^{<\mathbb{N}} \to 2^X$, called a Souslin operation, such that $I^\infty \setminus M = \bigcup_{\sigma \in \mathbb{N}^\mathbb{N}} \bigcap_{n=1}^\infty S(\sigma \restriction n)$ and $S(\tau) \subset \operatorname{int} S(\tau')$ for $\tau' < \tau$.

Put $\lambda_{\langle k \rangle} \equiv 1$ for all $k \in \mathbb{N}$. For $\tau \in \mathbb{N}^{<\mathbb{N}}$ with $|\tau| \geqslant 2$, let $\lambda_\tau : I^\infty \to [0,1]$ be a continuous function such that

$$\begin{aligned} &\lambda_\tau(q) = 1 \quad \text{for} \quad q \in S(\tau), \\ &\lambda_\tau(q) = 0 \quad \text{for} \quad q \notin \operatorname{int} S(\tau \restriction (|\tau| - 1)), \\ &0 < \lambda_\tau(q) < 1 \quad \text{for} \quad q \in \operatorname{int} S(\tau \restriction (|\tau| - 1)) \setminus S(\tau). \end{aligned} \tag{1}$$

Define a family $\mathcal{P} = \{P_\tau \subset I^2 : \tau \in \mathbb{N}^{<\mathbb{N}}\}$ of squares as follows.

Put
$$P_{\langle i \rangle} = [2^{-(2i-1)}, 2^{-2(i-1)}] \times [0, 2^{-(2i-1)}].$$

Suppose squares P_τ have been defined for all $\tau \in \mathbb{N}^{<\mathbb{N}}$ of length $\leq n$ such that

- one side of P_τ is contained in $I \times \{0\}$;
- if $\tau < \tau'$ then $P_{\tau'} \subset P_\tau$;
- if $|\tau| = |\tau'|$ and $\tau \neq \tau'$, then $P_{\tau'} \cap P_\tau = \emptyset$;
- $\operatorname{diam} P_\tau \leqslant 2^{-|\tau|}$.

Let $h_\tau : I^2 \to P_\tau$ be the similitude which maps $I \times \{0\}$ onto $P_\tau \cap (I \times \{0\})$ and let η be an element of $\mathbb{N}^{<\mathbb{N}}$ of length $n+1$. We have $\eta = \langle \eta \restriction n, j \rangle$, for some $j \in \mathbb{N}$. Put

$$P_\eta = h_{\eta \restriction n}(P_{\langle j \rangle}).$$

For each $\sigma \in \mathbb{N}^\mathbb{N}$ and natural number n, consider the mappings $\xi^n_\sigma : I^\infty \to C(I^\infty)$ and $\xi^n : I^\infty \to C(I^\infty)$ defined by

$$\xi^1_\sigma(q) = (I^k \times \{\mathbf{0}\}) \cup (P_{\sigma \restriction 1} \times [0,1] \times \{\mathbf{0}\}),$$

$$\xi^{n+1}_\sigma(q) = \xi^n_\sigma(q) \cup (P_{\sigma \restriction (n+1)} \times (\prod_{i=1}^{n+1} [0, \lambda_{\sigma \restriction i}(q)]) \times \{\mathbf{0}\}),$$

$$\xi^n(q) = \bigcup_{\sigma \in \mathbb{N}^\mathbb{N}} \xi^n_\sigma(q).$$

Observe that the mappings are continuous and

$$\xi^n(q) \subset \xi^{n+1}(q),$$

$$d_H(\xi^n(q), \xi^{n+1}(q)) \leq 2^{-(n+1)}.$$

Thus the sequence $(\xi^n)_{n\in\mathbb{N}}$ uniformly converges to a continuous mapping ξ such that

$$\xi(q) = (I^k \times \{\mathbf{0}\}) \cup \bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}} \bigcup_{i=1}^{\infty} (P_{\sigma\restriction i} \times \prod_{i=1}^{\infty} [0, \lambda_{\sigma\restriction i}(q)]). \tag{2}$$

If $q \in M$, then, for each $\sigma \in \mathbb{N}^{\mathbb{N}}$, the sequence $(\lambda_{\sigma\restriction n}(q))_{n\in\mathbb{N}}$ eventually equals 0, so the right hand side of (2) is the countable union of finite dimensional cubes. If $q \notin M$ then there is a sequence $\sigma \in \mathbb{N}^{\mathbb{N}}$ such that $q \in S(\sigma \restriction n)$ (and thus, $\lambda_{\sigma\restriction n}(q) = 1$) for each n. The diameters of squares $P_{\sigma\restriction n}$ tend to 0 if $n \to \infty$, so the intersection of the squares is a singleton $\{z\}$ and $\xi(q)$ contains a Hilbert cube of the form $\{z\} \times I^\infty$. Therefore $\xi^{-1}(\mathcal{SCD}_k(I^\infty)) = M$.

Now define the embedding $\theta : I^\infty \to C([-1,0] \times [-1,1] \times I^\infty)$. Denote by $O(a;r)$ the circle in $\mathbb{R}^2$ with center a and radius r. For each $q = (q_i) \in I^\infty$, put

$$r_i(q) = 4^{-(i+1)}(1 + q_i), \quad a_i = (-1 + 2^{-i}, 0) \in \mathbb{R}^2,$$

$$\theta_0(q) = ([-1,0] \times \{0\}) \cup O((-\frac{1}{2}, 0); \frac{1}{2}) \cup \bigcup_{i=1}^{\infty} O(a_i; r_i(q)) \subset [-1,0] \times [-1,1]$$

and $\theta(q) = \theta_0(q) \times \{\mathbf{0}\}$, where $\mathbf{0} = (0, 0, \dots) \in I^\infty$. The set $\theta(q)$ is the union of countably many mutually disjoint circles and of the diameter segment of the largest circle.

Observe that if $\mathcal{A} = \mathcal{SCD}_k(I^\infty)$ or $\mathcal{A} = \mathcal{SCD}_k \cap C(I^\infty)$, then $\mathcal{A}$ and the mappings ξ and θ satisfy the hypotheses of Lemma 3.2. □

Recently Krupski found new examples of coanalytic absorbers in 2^{I^∞} and $C(I^\infty)$.

Theorem 3.5 ([11])**.** *For each integer $n \geq 0$ denote by $\mathcal{W}_n$ the collection of weakly infinite-dimensional compacta of dimensions $\geq n$ in I^∞.*

1. *If $n \geq 1$ then $\mathcal{W}_n$ and $\mathcal{W}_n \cap C(I^\infty)$ are coanalytic absorbers in 2^{I^∞}.*
2. *If $n \geq 2$ then $\mathcal{W}_n \cap C(I^\infty)$ is a coanalytic absorber in $C(I^\infty)$.*

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